Gravitational Energy Of The Binary Black Hole Inspiral With An Application To GW151226

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Abstract

We investigate the gravitational energy-momentum distribution in the space-time of two black holes in circular orbit, in the context of the teleparallel equivalent of general relativity (TEGR). We address an approximate solution of Einstein's field equations that describes two non-spinning black holes that circle each other in the xy plane, obtain the total energy of the space-time, and verify that the gravitational binding energy is negative. For a fixed orbital separation the average value of the flux of gravitational radiation over a period vanishes, and we show that effective gravitational radiation is emitted as long as the separation between the holes decreases in time. We then illustrate the formalism with a concrete application to the gravitational-wave event GW151226: from the strain data of the L1 detector we reconstruct the time-dependent orbital parameters of the binary and evaluate numerically the total gravitational energy of the space-time, finding agreement with the analytical predictions.

Keywords: teleparallel gravity, gravitational energy, binary black holes, gravitational waves, GW151226

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1. INTRODUCTION

The physics of black hole coalescence has undergone a remarkable transformation since the first direct detection of gravitational waves by LIGO in 2015 [1]. Such signals are now routinely recorded by the LVK consortium, and the gravitational waves emitted by compact binary systems, including binary neutron stars, allow one to infer the physical characteristics and parameters of their sources [2]. In the past, post-Newtonian approximation methods were essentially the only tools available for theoretical progress in this field. However, since the first numerical simulation of the merger of two black holes in 2005 [3], the field has rapidly developed the techniques and methods of numerical relativity [4, 5]. Indeed, the LIGO strain data are processed through matched-filtering procedures that compare the recorded signal with templates produced by such simulations, in order to identify the gravitational-wave signal. It is worth emphasizing that, although the gravitational radiation is generated by the strongly nonlinear dynamics of the source, it reaches the detector in the regime of the linearized approximation; this distinction is conceptually important. Nevertheless, the radiation of mass (as energy), linear momentum and angular momentum remains an essential feature of the final stages of the merger, and leaves its imprint on the form of the detected signal [6]. In particular, the recoil (kick) acquired by the remnant after the merger may be physically relevant, and the conservation of the total linear momentum of the configuration implies that the linear momentum carried away by the field must compensate that of the binary/merged holes [7, 8, 9, 10].

It is well known that the energy, momentum and angular momentum radiated by a binary system are, within the metric formulation of general relativity, intrinsically global quantities: they are defined as surface integrals in the asymptotic region of the space-time, the ADM quantities at spatial infinity [11] or the Bondi-Sachs quantities at null infinity [12], while any local density constructed out of the metric reduces to a coordinate-dependent pseudotensor [13]. The teleparallel equivalent of general relativity (TEGR) is, by contrast, particularly well suited to the definition of conserved quantities of tensorial character: it furnishes a gravitational energy-momentum density that is well defined at each point of space, and which can therefore be integrated over an arbitrary finite volume [14, 15], together with the corresponding fluxes [16]. This local and integrable character is precisely what makes the TEGR an adequate framework in which to explore the exchange of energy and momentum between the gravitational field and its sources, specifically here, between the field and the black holes that constitute the binary. A natural arena for such an investigation is the orbital motion of two black holes in quasi-stationary, quasi-circular orbit: the conservation of the total linear momentum of the space-time requires the momentum of the gravitational field to balance that of the orbiting holes [17], so that a precise characterization of the field energy-momentum is indispensable in order to understand the flow of these quantities throughout the configuration. It is for this configuration that we derive, in the sections that follow, the total gravitational energy, the binding energy and the flux of gravitational radiation within the TEGR.

The expressions for the gravitational energy-momentum in the TEGR exhibit a definite transformation behaviour that we regard as a virtue rather than a shortcoming. They are invariant under coordinate transformations of the three-dimensional spacelike surface and under time reparametrizations, but they do depend on the frame of the observer, since P^a transforms as a vector under the global Lorentz group [18]. In particular, the gravitational energy is the zero component of this energy-momentum four-vector, and therefore possesses the standard transformation properties of the energy in special relativity. The energy-momentum of any physical system already depends on the frame of the observer in special relativity, and there is no compelling reason to drop this feature when one passes to general relativity, since there is a smooth transition

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between special and general relativity; here the frame is fixed by adopting observers that are static in the asymptotically flat space-time. In our opinion this approach is better suited to the present analysis than the one based on pseudotensors [17]. Pseudotensors are quantities that depend on the choice of the coordinates of the three-dimensional space, and are therefore not well behaved under coordinate transformations; the analysis of ref. [17], for instance, makes use of the Landau-Lifshitz pseudotensor [13], and we have never seen a justification as to why one pseudotensor should be preferred over another. The TEGR expressions, by contrast, are well-defined geometrical objects in the context of the Poincaré group, and it is on this basis that we proceed to analyse the binary system.

In this paper we adopt the approximate solution of Einstein's equations obtained by Alvi [19], later re-analysed and re-obtained in [20], which describes two nonspinning black holes in circular orbit, and we evaluate the corresponding gravitational energy-momentum of the space-time. In this approximate model the orbital motion of the holes is restricted to the xy plane, and the separation b between them is held fixed (in the context of ref. [19]). For two black holes with individual rest masses m_1 and m_2, we find that the time-averaged total energy $cP^{(0)}$ of the combined system in orbital motion is smaller than $c(P_1^{(0)} + P_2^{(0)}) = (m_1 + m_2)c^2$. The binding energy $E_b = cP^{(0)} - c(P_1^{(0)} + P_2^{(0)})$ is negative, has a remarkably simple form, and is in agreement with previous analyses. For a fixed value of b we obtain a simple expression for the total flux of gravitational radiation, whose average value over an orbital period vanishes. If, on the other hand, the separation decreases in time, $\dot{b} < 0$, the binding energy yields a positive flux of gravitational radiation. We thus conclude that effective gravitational radiation takes place provided $\dot{b} \neq 0$.

This paper is organized as follows. In section 2 we review the formulation of the TEGR, and show how the definition of the gravitational energy-momentum arises out of the field equations of the theory; we also present the expressions for the fluxes of gravitational radiation and of matter fields. In section 3 we describe the approximate solution for two black holes in circular orbit, and in section 4 we evaluate the energy and momentum of the gravitational field, obtaining simple expressions for the total gravitational energy, the binding energy and the total flux of gravitational radiation. In section 5 we illustrate the formalism with a concrete application to the gravitational-wave event GW151226: using publicly available strain data from the L1 detector, we reconstruct the time-dependent orbital parameters of the binary and evaluate numerically the total gravitational energy of the space-time, showing that the numerical results are consistent with the analytical predictions of section 4. Finally, we present our conclusions in section 6.

Notation: space-time indices $\mu, \nu, \ldots$ and SO(3,1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The tetrad field is denoted $e^a{}_\mu$, and the torsion tensor reads $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. The flat, Minkowski spacetime metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-1, +1, +1, +1)$. The determinant of the tetrad field is represented by $e = \det(e^a{}_\mu)$.

2. ENERGY-MOMENTUM IN THE TEGR

In the teleparallel equivalent of general relativity the gravitational field is represented by the tetrad field $e^a{}_\mu$ only, and the Lagrangian density is written in terms of the torsion tensor $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. This tensor is related to the antisymmetric part of the Weitzenböck connection $\Gamma^\lambda_{\mu\nu} = e^{a\lambda} \partial_\mu e_{a\nu}$. However, the dynamics of the gravitational field in the TEGR is essentially the same as in the usual metric formulation. The physics in both formulations is identically the same.

Let us start with the torsion-free, Levi-Civita connection ${}^0\omega_{\mu ab}$,

$$\begin{aligned} {}^0\omega_{\mu ab} &= -\frac{1}{2} e^c{}_\mu (\Omega_{abc} - \Omega_{bac} - \Omega_{cab}) , \\ \Omega_{abc} &= e_{a\nu} (e_b{}^\mu \partial_\mu e_c{}^\nu - e_c{}^\mu \partial_\mu e_b{}^\nu) . \end{aligned} \tag{1}$$

The Christoffel symbols ${}^0\Gamma^\lambda_{\mu\nu}$ and the Levi-Civita connection are identically related by

$${}^0\Gamma^\lambda_{\mu\nu} = e^{a\lambda} \partial_\mu e_{a\nu} + e^{a\lambda} ({}^0\omega_{\mu ab}) e^b{}_\nu .$$

In view of this expression an identity arises between the Levi-Civita connection and the contorsion tensor $K_{\mu ab}$,

$${}^0\omega_{\mu ab} = -K_{\mu ab} , \tag{2}$$

where $K_{\mu ab} = \frac{1}{2} e_a{}^\lambda e_b{}^\nu (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} + T_{\mu\lambda\nu})$, and $T_{\lambda\mu\nu} = e^a{}_\lambda T_{a\mu\nu}$. Making use of eq. (2) it follows that the scalar curvature $R(e)$ may be identically written as

$$eR({}^0\omega) = -e \left(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) + 2\partial_\mu (eT^\mu) , \tag{3}$$

where e is the determinant of the tetrad field. Therefore in the framework of the TEGR the Lagrangian density for the gravitational and matter fields is defined by

$$\begin{aligned} L &= -ke(\frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a) - \frac{1}{c} L_M \\ &\equiv -ke \Sigma^{abc} T_{abc} - \frac{1}{c} L_M , \end{aligned} \tag{4}$$

where $k = c^3/16\pi G$, $T_a = T^b{}_{ba}$, $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$ and

$$\Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^b - \eta^{ab} T^c) . \tag{5}$$

L_M stands for the Lagrangian density for the matter fields. The Lagrangian density L is invariant under the global SO(3,1) group. The absence in the Lagrangian density of the divergence term on the right hand side of eq. (3) prevents the invariance of (4) under arbitrary local SO(3,1) transformations.

The field equations derived from (4) are equivalent to Einstein's equations. They read

$$e_{a\lambda} e_{b\mu} \partial_\nu (e \Sigma^{b\lambda\nu}) - e(\Sigma^{b\nu}{}_a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd}) = \frac{1}{4kc} e T_{a\mu} , \tag{6}$$

where $\delta L_M / \delta e^{a\mu} = e T_{a\mu}$. From now on we will make $c = 1 = G$.

The definition of the gravitational energy-momentum may be established in the framework of the Lagrangian formulation defined by (4), according to the procedure of ref. [15]. Equation (6) may be rewritten as

$$\partial_\nu(e\Sigma^{a\lambda\nu}) = \frac{1}{4k} e\, e^a{}_\mu (t^{\lambda\mu} + T^{\lambda\mu}) \,, \tag{7}$$

where $T^{\lambda\mu} = e_a{}^\lambda T^{a\mu}$ and $t^{\lambda\mu}$ is defined by

$$t^{\lambda\mu} = k(4\Sigma^{bc\lambda} T_{bc}{}^\mu - g^{\lambda\mu}\Sigma^{bcd} T_{bcd}) \,. \tag{8}$$

In view of the antisymmetry property $\Sigma^{a\mu\nu} = -\Sigma^{a\nu\mu}$ it follows that

$$\partial_\lambda \left[e\, e^a{}_\mu (t^{\lambda\mu} + T^{\lambda\mu}) \right] = 0 \,. \tag{9}$$

The equation above yields the continuity (or balance) equation,

$$\frac{d}{dt} \int_V d^3x\, e\, e^a{}_\mu (t^{0\mu} + T^{0\mu}) = - \oint_S dS_j \, \left[e\, e^a{}_\mu (t^{j\mu} + T^{j\mu}) \right] \,, \tag{10}$$

Therefore we identify $t^{\lambda\mu}$ as the gravitational energy-momentum tensor [15],

$$P^a = \int_V d^3x\, e\, e^a{}_\mu (t^{0\mu} + T^{0\mu}) \,, \tag{11}$$

as the total energy-momentum contained within a volume V of the three-dimensional space,

$$\Phi^a_g = \oint_S dS_j \, (e\, e^a{}_\mu t^{j\mu}) \,, \tag{12}$$

as the gravitational energy-momentum flux [16], and

$$\Phi^a_m = \oint_S dS_j \, (e\, e^a{}_\mu T^{j\mu}) \,, \tag{13}$$

as the energy-momentum flux of matter. In view of (7) eq. (11) may be written as

$$P^a = - \int_V d^3x \partial_j \Pi^{aj} = - \oint_S dS_j \, \Pi^{aj} \,, \tag{14}$$

where $\Pi^{aj} = -4ke\,\Sigma^{a0j}$. The expression above is the definition for the gravitational energy-momentum presented in ref. [14], obtained in the framework of the vacuum field equations in Hamiltonian form. It is invariant under coordinate transformations of the three-dimensional space and under time reparametrizations. We note that (9) is a true energy-momentum conservation equation.

Finally we remark that in the absence of matter fields the total flux of gravitational radiation $\Phi^{(0)}$ is related to the total gravitational energy according to $\Phi^{(0)} = -\dot{P}^{(0)}$, in view of eq. (10).

3. THE SPACE-TIME OF THE BINARY BLACK HOLE

We consider the approximate metric for the binary black hole as described in ref. [19]. This solution was later re-analyzed and re-obtained in ref. [20] by matching two perturbed Schwarzschild metrics to an asymptotically post-Newtonian construction for a binary black hole space-time. The two black holes have masses m_1 and m_2, and circle around each other in the plane xy. We restrict the analysis to the coordinates in the radiation zone, defined by $r >> \lambda_{GW}$ [21, 22], where λ_{GW} is the wavelength of the gravitational radiation. In the context of the present analysis the radiation zone is established in an equivalent way by $r >> m_1$, $r >> m_2$ and $r >> b$ (see below the definition of b). We define

$$m = m_1 + m_2 \,, \quad \delta m = m_1 - m_2 \,, \quad \mu = \frac{m_1 m_2}{m} \,. \tag{15}$$

The circular, Newtonian trajectories of the black holes are

$$\mathbf{r}_1(t) = \frac{m_2}{m}\mathbf{b}(t) \,, \quad \mathbf{r}_2(t) = -\frac{m_1}{m}\mathbf{b}(t) \,, \tag{16}$$

where

$$\mathbf{b}(t) = \mathbf{r}_1(t) - \mathbf{r}_2(t) = b(\cos\omega t, \sin\omega t, 0) \,, \tag{17}$$

and

$$\omega = \sqrt{\frac{m}{b^3}} \,,$$

is the orbital angular velocity. The separation b is defined by $b = |\mathbf{r}_1 - \mathbf{r}_2|$. The velocities of the holes are

$$\mathbf{v}_1 = \frac{d\mathbf{r}_1}{dt} \,, \quad \mathbf{v}_2 = \frac{d\mathbf{r}_2}{dt} \,, \tag{18}$$

from what we define

$$\mathbf{v}(t) = \mathbf{v}_1 - \mathbf{v}_2 = \sqrt{\frac{m}{b}}(-\sin\omega t, \cos\omega t, 0) \,. \tag{19}$$

In the radiation zone the metric components depend on $\mathbf{b}$ and $\mathbf{v}$, which in turn depend not exactly on t, but on the retarded time $\tau = t - r$ [19]. We further define

$$\tilde{m} = m\left(1 - \frac{\mu}{2b}\right) \,, \quad \mathbf{n} = \frac{\mathbf{r}}{r} \,, \tag{20}$$

where $\mathbf{r}$ is a point of observation in space.

With the help of the definitions above the metric tensor components for the binary black hole space-time in the radiation zone read [19]

$$
\begin{aligned}
g_{00} &= -1+\frac{2\tilde{m}}{r}-\frac{2\tilde{m}^2}{r^2} \\
&\quad +\frac{\mu}{r}\left\{2(\mathbf{n}\cdot\mathbf{v})^2-\frac{2m}{b^3}(\mathbf{n}\cdot\mathbf{b})^2+\frac{6}{r}(\mathbf{n}\cdot\mathbf{b})(\mathbf{n}\cdot\mathbf{v})+\frac{1}{r^2}[3(\mathbf{n}\cdot\mathbf{b})^2-b^2]\right\} \\
&\quad +\frac{\mu}{r}\frac{\delta m}{m}\left\{(\mathbf{n}\cdot\mathbf{v})\left[\frac{7m}{b^3}(\mathbf{n}\cdot\mathbf{b})^2-2(\mathbf{n}\cdot\mathbf{v})^2-\frac{m}{b}\right]\right. \\
&\quad +\frac{2}{r}(\mathbf{n}\cdot\mathbf{b})\left[\frac{3m}{b^3}(\mathbf{n}\cdot\mathbf{b})^2-6(\mathbf{n}\cdot\mathbf{v})^2-\frac{m}{b}\right] \\
&\quad \left.+\frac{3}{r^2}(\mathbf{n}\cdot\mathbf{v})[b^2-5(\mathbf{n}\cdot\mathbf{b})^2]+\frac{1}{r^3}(\mathbf{n}\cdot\mathbf{b})[3b^2-5(\mathbf{n}\cdot\mathbf{b})^2]\right\}, \\
g_{0i} &= -\frac{4\mu}{r}\left\{\left[(\mathbf{n}\cdot\mathbf{v})+\frac{1}{r}(\mathbf{n}\cdot\mathbf{b})\right]v^i-\frac{m}{b^3}(\mathbf{n}\cdot\mathbf{b})b^i\right\} \\
&\quad +\frac{2\mu}{r}\frac{\delta m}{m}\left(\left\{2(\mathbf{n}\cdot\mathbf{v})^2-\frac{3m}{b^3}(\mathbf{n}\cdot\mathbf{b})^2+\frac{6}{r}(\mathbf{n}\cdot\mathbf{b})(\mathbf{n}\cdot\mathbf{v})\right.\right. \\
&\quad \left.+\frac{1}{r^2}[3(\mathbf{n}\cdot\mathbf{b})^2-b^2]\right\}v^i \\
&\quad \left.+\left\{-\frac{4m}{b^3}(\mathbf{n}\cdot\mathbf{b})(\mathbf{n}\cdot\mathbf{v})+\frac{m}{rb}\left[1-\frac{3}{b^2}(\mathbf{n}\cdot\mathbf{b})^2\right]\right\}b^i\right), \\
g_{ij} &= \delta_{ij}\left(1+\frac{2\tilde{m}}{r}+\frac{m^2}{r^2}\right. \\
&\quad +\frac{\mu}{r}\left\{2(\mathbf{n}\cdot\mathbf{v})^2-\frac{2m}{b^3}(\mathbf{n}\cdot\mathbf{b})^2+\frac{6}{r}(\mathbf{n}\cdot\mathbf{b})(\mathbf{n}\cdot\mathbf{v})+\frac{1}{r^2}[3(\mathbf{n}\cdot\mathbf{b})^2-b^2]\right\} \\
&\quad +\frac{\mu}{r}\frac{\delta m}{m}\left\{(\mathbf{n}\cdot\mathbf{v})\left[\frac{7m}{b^3}(\mathbf{n}\cdot\mathbf{b})^2-2(\mathbf{n}\cdot\mathbf{v})^2+\frac{m}{b}\right]\right. \\
&\quad +\frac{6}{r}(\mathbf{n}\cdot\mathbf{b})\left[\frac{m}{b^3}(\mathbf{n}\cdot\mathbf{b})^2-2(\mathbf{n}\cdot\mathbf{v})^2\right] \\
&\quad \left.\left.+\frac{3}{r^2}(\mathbf{n}\cdot\mathbf{v})[b^2-5(\mathbf{n}\cdot\mathbf{b})^2]+\frac{1}{r^3}(\mathbf{n}\cdot\mathbf{b})[3b^2-5(\mathbf{n}\cdot\mathbf{b})^2]\right\}\right) \\
&\quad +\frac{m^2}{r^2}n^in^j+\frac{4\mu}{r}\left(v^iv^j-\frac{m}{b^3}b^ib^j\right)+\frac{2\mu}{r}\frac{\delta m}{m}\left\{\frac{6m}{b^3}(\mathbf{n}\cdot\mathbf{b})v^{(i}b^{j)}\right. \\
&\quad \left.+\left[(\mathbf{n}\cdot\mathbf{v})+\frac{1}{r}(\mathbf{n}\cdot\mathbf{b})\right]\left(\frac{m}{b^3}b^ib^j-2v^iv^j\right)\right\}. \qquad (21)
\end{aligned}
$$

Our aim is to evaluate definitions (14) and (12) for the gravitational energy-momentum and the corresponding fluxes. These definitions are invariant under global SO(3,1) transformations. Therefore they are frame dependent. However, P^a is a vector under global Lorentz transformations. The frame dependence of the gravitational energy-momentum is understood by simply considering a black hole of mass m and an observer that is very distant from the black hole. The black hole will appear to this observer as a particle of mass m, with energy $E = cP^{(0)} = mc^2$ (m is the mass of the black hole in the frame where the black hole is at rest). If, however, the black hole is moving at velocity v with respect to the observer, then its total gravitational energy will be $E = \gamma mc^2$, where $\gamma = (1 - v^2/c^2)^{-1/2}$. This example is a consequence of the special theory of relativity, and shows clearly the frame dependence of the gravitational energy-momentum. The frame dependence is not restricted to observers at spacelike infinity. It holds for observers located everywhere in the three-dimensional space.

In order to evaluate definitions (11–14) out of the metric tensor given by (21) we choose a configuration of tetrad fields that has a clear physical interpretation. In the framework of the TEGR the tetrad field describes both the gravitational field and the frame. For a given metric tensor there exists an infinity of possible frames, and each frame is characterized by six conditions on the tetrad field. Three conditions fix the kinematical state of the observer in the three-dimensional space (for instance, the observer may be stationary in space), and the other three conditions fix the orientation of the frame (alternatively, the frame may be characterized by the six components of the acceleration tensor ϕ_{ab} [18]).

Therefore tetrad fields are interpreted as reference frames adapted to preferred fields of observers in spacetime. This interpretation is possible by identifying the ${e_{(0)}}^{\mu}$ components of the frame with the four-velocities u^μ of the observers, ${e_{(0)}}^{\mu} = u^\mu$ [18]. Here we will establish a set of tetrad fields adapted to static observers in spacetime. Thus we require ${e_{(0)}}^{i} = 0$. This condition fixes 3 components of the frame. The other three components are fixed by choosing an orientation of the frame in the three-dimensional space. Therefore ${e_{(0)}}^{\mu}$ is parallel to the worldline of the observers, and ${e_{(k)}}^{\mu}$ are the three unit vectors orthogonal to the timelike direction. We fix ${e_{(k)}}^{\mu}$ such that ${e_{(1)}}^{\mu}$, ${e_{(2)}}^{\mu}$ and ${e_{(3)}}^{\mu}$ in Cartesian Coordinates (and in the flat space-time limit) are unit vectors along the x, y and z directions. The tetrad field that satisfies these conditions is given by

$$ {e^a}_{\mu}(t,x,y,z) = \begin{pmatrix} A & B & C & 0 \\ 0 & D & E & F \\ 0 & 0 & G & H \\ 0 & 0 & 0 & I \end{pmatrix} , \qquad (22) $$

with the following definitions:

$$ \begin{aligned} A &= (-g_{00})^{1/2} , \\ B &= -\frac{g_{01}}{(-g_{00})^{1/2}} , \\ C &= -\frac{g_{02}}{(-g_{00})^{1/2}} , \\ D &= \frac{\lambda_{11}}{(-g_{00})^{1/2}} , \\ E &= \frac{1}{(-g_{00})^{1/2}} \frac{\lambda_{12}^2}{\lambda_{11}} , \\ F &= (-g_{00})^{1/2} \frac{g_{13}}{\lambda_{11}} , \\ G &= \frac{1}{(-g_{00})^{1/2}} \left[\lambda_{22}^2 - \frac{\lambda_{12}^4}{\lambda_{11}^2} \right]^{1/2} , \\ H &= \frac{(-g_{00})^{1/2}}{\lambda_{11}} \frac{g_{23}\lambda_{11}^2 - g_{13}\lambda_{12}^2}{(\lambda_{11}^2\lambda_{22}^2 - \lambda_{12}^4)^{1/2}} , \\ I &= \frac{1}{\lambda_{11}} \left[g_{33}\lambda_{11}^2 - (-g_{00}) \left(g_{13}^2 + \frac{(g_{23}\lambda_{11}^2 - g_{13}\lambda_{12}^2)^2}{\lambda_{11}^2\lambda_{22}^2 - \lambda_{12}^4} \right) \right]^{1/2} \end{aligned} \qquad (23) $$

The quantity λ_{ij} is defined by $\lambda_{ij}^2 = g_{0i}g_{0j} - g_{00}g_{ij}$, and all metric components are obtained from (21). In the limit $r \to \infty$ the asymptotic quantities h_{00}, h_{11}, h_{22} and h_{33} are defined by the expressions

$$ \begin{aligned} g_{00} &= -1 + h_{00} , \\ g_{11} &= 1 + h_{11} , \\ g_{22} &= 1 + h_{22} , \\ g_{33} &= 1 + h_{33} . \end{aligned} \qquad (24) $$

In terms of these quantities the asymptotic form of the tetrad field is reduced to

$$ {e^a}_{\mu}(t,x,y,z) \cong \begin{pmatrix} 1 - \frac{h_{00}}{2} & -g_{01} & -g_{02} & 0 \\ 0 & 1 + \frac{h_{11}}{2} & g_{12} & g_{13} \\ 0 & 0 & 1 + \frac{h_{22}}{2} & g_{23} \\ 0 & 0 & 0 & 1 + \frac{h_{33}}{2} \end{pmatrix} . \qquad (25) $$

Expression (22) represents a frame that is adapted to static observers everywhere in space-time.

4. GRAVITATIONAL ENERGY OF BINARY BLACK HOLES IN CIRCULAR MOTION

For a given space-time metric tensor and a given frame, the energy-momentum of the space-time is evaluated out of eq. (14). It reads

$$ P^a = 4k \oint_S dS_j \, e\Sigma^{a0j} . \qquad (26) $$

If the surface of integration S is fixed at spatial infinity, i.e., $S \to \infty$, P^a yields the total energy-momentum of the space-time. The latter is the same for all tetrad fields that exhibit the same

asymptotic behaviour. In particular, the energy-momentum obtained out of frames that are adapted to static observers at spacelike infinity coincides with the one obtained out of (22).

Considering the tetrad field given by eq. (22), the gravitational energy of the space-time determined by eq. (21) is given by

$$\begin{aligned} P^{(0)} &= 4k \oint_S dS_j \, e(e^{(0)}{}_0 \Sigma^{00j} + e^{(0)}{}_1 \Sigma^{10j} + e^{(0)}{}_2 \Sigma^{20j}) \\ &= 4k \oint_S dS_j \, e(A\,\Sigma^{00j} + B\,\Sigma^{10j} + C\,\Sigma^{20j}) \,, \end{aligned} \tag{27}$$

where A, B and C are defined by (23), $\Sigma^{\mu\nu\lambda}$ is calculated out of (5), and $j = 1, 2, 3$. We will evaluate the expression of $P^{(0)}$ for a closed surface S in the asymptotic region $r >> m$ and $r >> b$, which characterizes the radiation zone, and then we make $S \to \infty$. The metric tensor may be decomposed as $g_{\mu\nu} \cong \eta_{\mu\nu} + h_{\mu\nu}$, and $h_{\mu\nu}$ is of order $1/r$ at spacelike infinity. Thus we also have $g^{\mu\nu} \cong \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. In view of the intricate structure of the metric tensor given by (21), in the evaluation of (27) we will make the approximation $\Sigma^{\mu\nu\lambda} \cong \eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\lambda\gamma}\Sigma_{\alpha\beta\gamma}$, making sure that $e\Sigma^{a0j}$ is of order (at least) $1/r^2$. After a large number of calculations, and taking into account the approximations explained above, we obtain the exact expression for the total energy of the space-time,

$$\begin{aligned} P^{(0)} &= (m_1 + m_2)\left[1 - \frac{m_1 m_2}{2(m_1+m_2)b}\right] \\ &- \left\{\frac{8}{3}\frac{\mu^2 m^2}{b^2}\omega\cos(2\omega\tau) + \frac{\mu^2(\delta m)^2}{b^2}\left(\frac{m\omega}{b}\right)\left[\frac{3}{2}\cos(2\omega\tau)\right.\right. \\ &+ \left.\left.\frac{3}{8}\cos^4\omega\tau + \frac{29}{8}\sin^4\omega\tau\right]\right\}\sin\omega\tau\cos\omega\tau \,. \end{aligned} \tag{28}$$

Assuming that the energy-momentum of matter fields vanishes for the binary black holes, then the expression above does represent the gravitational energy of the space-time.

We will be interested in average values of time dependent quantities. Therefore we define

$$< P^{(0)} >= \frac{1}{T}\int_0^T d\tau \, P^{(0)} \,, \tag{29}$$

where $T = 2\pi/\omega$. We easily obtain

$$< P^{(0)} >= (m_1 + m_2)\left[1 - \frac{m_1 m_2}{2(m_1+m_2)b}\right] . \tag{30}$$

We define the binding energy of the configuration according to $E_b =< P^{(0)} > -(m_1 + m_2)$. We find

$$E_b = -\frac{m_1 m_2}{2b} \,. \tag{31}$$

We note that E_b is the standard non-spinning part of the expression for the binding gravitational energy [23, 24] for two black holes in circular orbit. This term is precisely the same as the first term in eq. (191) of ref. [25].

We evaluate now the total flux $\Phi^{(0)} = -\dot{P}^{(0)}$ of gravitational radiation. After a large number of calculations we arrive at

$$\begin{aligned} \Phi^{(0)} &= \left\{\frac{8}{3}\frac{\mu^2 m^2}{b^2}\omega^2\cos(2\omega\tau) + \frac{\mu^2(\delta m)^2}{b^2}\frac{m\omega^2}{b}\left[\frac{3}{2}\cos(2\omega\tau)\right.\right. \\ &+ \left.\left.\frac{3}{8}\cos^4\omega\tau + \frac{29}{8}\sin^4\omega\tau\right]\right\}\cos(2\omega\tau) \\ &+ \left\{-\frac{16}{3}\frac{\mu^2 m^2}{b^2}\omega^2\sin(2\omega\tau)\right. \\ &+ \frac{\mu^2(\delta m)^2}{b^2}\frac{m\omega^2}{b}\left[-3\sin(2\omega\tau) - \frac{3}{2}\cos^3\omega\tau\sin\omega\tau\right. \\ &+ \left.\left.\frac{29}{2}\sin^3\omega\tau\cos\omega\tau\right]\right\}\sin\omega\tau\cos\omega\tau \,. \end{aligned} \tag{32}$$

An interesting consequence of the expression above is that the average value of $\Phi^{(0)}$ over a complete cycle vanishes,

$$< \Phi^{(0)} >= 0 \,. \tag{33}$$

We conclude that the orbital (stationary) motion of two black holes (or two point masses) on a plane produces the gravitational radiation given by eq. (32). However, the average value of this radiation vanishes. On the other hand, the situation changes if the separation distance b changes with time.

In order to obtain (32) we have considered that b is not a function of t. This condition was assumed in ref. [19]. According to ref. [20], however, this condition may be relaxed. Thus we may admit that b varies slowly with time. Taking into account this assumption we obtain an expression for the total flux of the gravitational radiation. For arbitrary $b(t)$ we have $\dot{\omega} \neq 0$. Consequently neither a complete cycle nor a period can be defined. In fact, the concept of average value does not apply to this case. $\Phi^{(0)}$ is now clearly nonvanishing. We present here only the contribution to $\Phi^{(0)}$ of the variation in time of the first term on the right hand side of eq. (28), namely, the variation in time of the binding energy, which does not even contribute to (32). We find

$$\Phi^{(0)} \approx -\frac{dE_b}{dt} = -\frac{m_1 m_2}{2b^2}\dot{b} \,. \tag{34}$$

This flux will be positive definite provided $\dot{b} < 0$, which is the case for realistic binary black holes prior to the merger.

We have evaluated the momentum component $P^{(3)}$, which is the component oriented along the z direction. The integration is carried out over a finite cubic volume with sides a, such that $a >> b$. We found that $P^{(3)} = 0$. This result is expected since the black holes are restricted to the xy plane, and there is no flux of momentum along the z direction. When the integration is carried out over the whole three-dimensional space, the total momentum of the space-time vanishes, i.e., $P^{(i)} = 0$ for $i = 1, 2, 3$.

5. NUMERICAL EVALUATION OF THE GRAVITATIONAL ENERGY FOR GW151226

The previous sections developed the formal expressions for the gravitational energy and momentum of a binary black hole space-time within the framework of the TEGR. The analytical results of section 4, in particular eqs. (28), (30) and (31) for the total energy, the mean binding energy and the radiation flux, were obtained for the post-Newtonian metric of Alvi [19],

which describes two non-spinning black holes in quasi-circular orbit in the radiation zone. This metric is valid during the inspiral phase, when the orbital separation b is much larger than the individual horizons but the system is still well described by a slowly evolving Keplerian binary.

It is therefore natural to ask whether the formalism can be applied to an actual astrophysical event observed during this inspiral regime. In this section we carry out such an application, using publicly available data from the gravitational wave event GW151226. The aim is twofold: to illustrate the formalism with realistic input data, and to verify the compatibility between the numerical evaluation of eq. (27) and the analytical results of section 4, evaluated at the orbital parameters reconstructed from the measured strain.

The event GW151226 was detected by LIGO on December 26, 2015, and was the second confirmed observation of gravitational waves from a binary black hole merger [26]. The source consists of two stellar-mass black holes with component masses $m_1 = 14.2\,M_\odot$ and $m_2 = 7.5\,M_\odot$, located at a luminosity distance of ~ 440 Mpc. The published analysis indicates a small but non-vanishing effective spin, $\chi_{\rm eff} \simeq 0.2$; since the metric of ref. [19] is non-spinning, we are implicitly neglecting this contribution, which is a controlled approximation in the inspiral regime considered here. The detection covers approximately 1 s of inspiral, during which the gravitational-wave frequency sweeps from about 35 to 450 Hz, before the merger and ringdown. The inspiral portion of this signal lies entirely within the regime of validity of the metric of ref. [19]: the orbital separation decreases from ~ 650 to ~ 450 km, while the individual horizons remain of order tens of kilometers, so that the hierarchy $b \gg m_1, m_2$ is preserved throughout. This makes GW151226 a particularly suitable event for the present analysis.

The strain data from the L1 detector were obtained from the Gravitational Wave Open Science Center [27], which provides public access to calibrated time series from the LIGO and Virgo observatories. From this strain we reconstruct the time-dependent metric tensor entering eq. (27), and evaluate numerically the gravitational energy $P^{(0)}(t)$ over the inspiral window. The methodology and results are presented in the two subsections that follow.

5.1. From the L1 strain to the time-dependent metric

The procedure consists of three steps: (i) acquiring and preprocessing the strain data from the L1 detector, (ii) fitting a theoretical waveform to the cleaned strain in order to recover the phase evolution of the signal, and (iii) reconstructing from this phase the time-dependent orbital parameters that enter the metric tensor (21).

The strain time series is obtained from the GWOSC archive for the L1 detector, with a sampling rate of 4096 Hz and a duration of 32 s centered at the event GPS time. The data are then mean-subtracted and passed through a chain of notch filters that suppress the known instrumental lines of the detector, including the calibration lines and the harmonics of the power grid. A Butterworth band-pass filter is subsequently applied in the band $[30, 500]$ Hz, which contains essentially all of the GW151226 signal power. In parallel, a whitened version of the strain is produced; this version is used only to locate the merger time $t_{\rm merger}$ through the maximum of its analytic envelope, and not in the subsequent fit.

The cleaned strain is then compared with a theoretical inspiral waveform. We employ the IMRPhenomD frequency-domain approximant [28], evaluated in the time domain with the component masses $m_1 = 14.2\,M_\odot$ and $m_2 = 7.5\,M_\odot$ of GW151226, at the luminosity distance of 440 Mpc. The approximant returns the two polarisations $h_+(t)$ and $h_\times(t)$. We restrict the fit to an inspiral window of duration 1.2 s, ending 50 ms before $t_{\rm merger}$, in order to remain safely inside the regime of validity of the metric (21). Within this window we adjust three coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the linear combination

$$h_{\rm fit}(t) = \mathcal{A}\,h_+(t-\tau) + \mathcal{B}\,h_\times(t-\tau) + \mathcal{C}\,, \tag{35}$$

together with a global time delay τ, by minimising the squared residual with respect to the cleaned strain. The delay τ is scanned over a window of ± 0.25 s, and for each value of τ the coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are obtained from a linear least-squares step. The best-fit parameters yield $\tau = +0.229$ s, $\mathcal{A} = 5.54$, $\mathcal{B} = 0.80$, with $\mathcal{C}$ consistent with zero ($\sim 10^{-24}$). The corresponding fit is shown in Fig. 1.

From the best-fit polarisations one constructs the complex signal $z(t) = h_+(t) - i\,h_\times(t)$ and extracts the gravitational-wave phase as $\phi_{\rm gw}(t) = \arg z(t)$, unwrapped along the inspiral window. The instantaneous gravitational-wave frequency is then $\omega_{\rm gw}(t) = d\phi_{\rm gw}/dt$, and the orbital angular frequency follows from $\omega(t) = \omega_{\rm gw}(t)/2$. A short Savitzky-Golay filter is used to suppress numerical noise in the derivative without altering the global behaviour of $\omega(t)$. The orbital separation $b(t)$ is then recovered from the Keplerian relation given in section 3, with ω now reconstructed from the data instead of being held constant. The orbital phase is obtained by integrating $\omega(t)$, and the radial velocity $\dot b(t)$ by analytic differentiation. Fig. 2 displays $f_{\rm gw}(t) = \omega_{\rm gw}(t)/(2\pi)$ together with $b(t)$ over the inspiral window.

With $b(t)$, $\omega(t)$, the orbital phase, and $\dot b(t)$ in hand, all the kinematical quantities entering the metric (21) are fully specified at each instant, namely the relative position $\mathbf{b}(t)$ defined in (17), the relative velocity $\mathbf{v}(t)$ defined in (19), and the unit vector $\mathbf{n}$ at any point of observation. The reconstructed metric is therefore available throughout the inspiral window as a time-dependent field on the three-dimensional space, ready to be used in the evaluation of the gravitational energy in the next subsection.

5.2. Gravitational energy of GW151226

The total gravitational energy of the space-time, $E(t) = cP^{(0)}(t)$, is computed from eq. (27) by direct numerical integration. The surface of integration is a cube of side $L = 440$ Mpc centered at the binary, with each face discretised by a regular two-dimensional grid. At each instant of the inspiral window, the metric tensor (21) is evaluated on the grid using the reconstructed orbital parameters $b(t)$, $\omega(t)$, $\dot b(t)$ and the orbital phase obtained in the previous subsection, and the spatial and temporal derivatives of the metric entering the integrand of eq. (27) are computed analytically before being summed over the six faces of the cube. The result is shown in Fig. 3.

The structure of the integrand of eq. (27) naturally separates the gravitational energy into two physically distinct contributions, since the tensor Σ^{a0j} is constructed out of first derivatives of the tetrad field, and hence of the metric tensor. The terms involving the spatial derivatives $\partial_i g_{\mu\nu}$ give rise to a piece that is large in magnitude and slowly varying in time; the terms

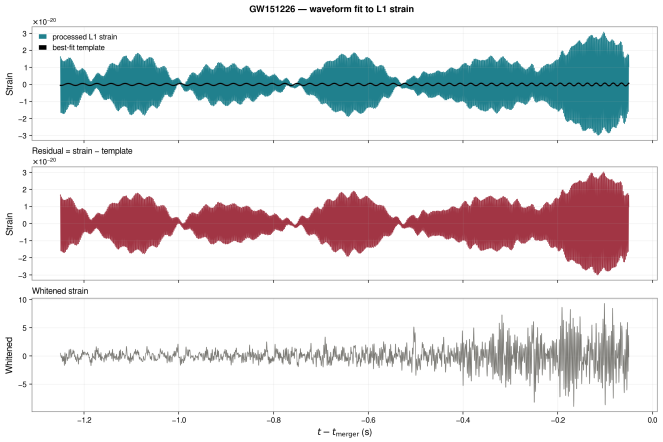

FIGURE 1: Top: processed L1 strain (blue) and best-fit IMRPhenomD template (black) over the inspiral window $t - t_{\rm merger} \in [-1.25, -0.05]$ s. Middle: residual strain = data − template. Bottom: whitened strain, used only to locate the merger epoch through the maximum of its analytic envelope.

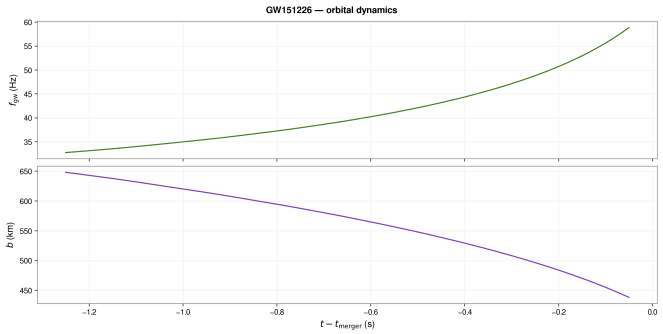

FIGURE 2: Orbital dynamics of GW151226 reconstructed from the best-fit template. Top: gravitational-wave frequency $f_{\rm gw}(t) = \omega_{\rm gw}(t)/(2\pi)$. Bottom: orbital separation $b(t)$, which decreases from approximately 650 to 440 km over the inspiral window.

involving the temporal derivative $\partial_0 g_{\mu\nu}$ give rise to a piece that is much smaller in magnitude and oscillates rapidly at the gravitational-wave frequency. We denote the latter by $\Delta E(t)$ and interpret it as the gravitational-radiation contribution to the total energy: its time average tends to zero, in agreement with eq. (33).

The slowly-varying piece can be further decomposed into a constant part and a slowly time-dependent part. The constant part is the ADM mass-energy of the configuration, $mc^2 = 3.878 \times 10^{48}$ J, which arises from the asymptotic $2\tilde{m}/r$ behaviour of the metric components in eq. (21) and corresponds to the energy of two isolated black holes at rest. The slowly time-dependent part is the binding energy $E_b(t)$ given by eq. (31), evaluated at the reconstructed $b(t)$. The full decomposition reads

$$E(t) = mc^2 + E_b(t) + \Delta E(t)\,, \tag{36}$$

where the first term dominates the total energy budget, the second captures the inspiral-driven energy loss, and the third encodes the gravitational radiation.

The upper panel of Fig. 3 displays the sum $mc^2 + E_b(t)$, with $\Delta E(t)$ invisible at the scale of 10^{48} J. The curve lies systematically below the dashed line mc^2, as expected from the negative sign of E_b in eq. (31). Numerically, $E - mc^2$ varies from $\approx -2.3 \times 10^{46}$ J at the beginning of the window to $\approx -3.3 \times 10^{46}$ J near the merger, consistent with eq. (31) evaluated at $b \approx 650$ km and $b \approx 440$ km respectively. The monotonic decrease of E along the inspiral reflects the gradual tightening of the binding as $b(t)$ shrinks, and corresponds to the gravitational radiation flux of eq. (34). The energy lost over the 1.2 s inspiral window is of order 10^{46} J.

The lower panel of Fig. 3 displays $\Delta E(t)$ isolated at its own scale of 10^{42} J, six orders of magnitude below the dominant contribution. The oscillation frequency follows the gravitational-wave frequency $f_{\rm gw}(t)$ of Fig. 2, growing from ~ 35 Hz at the start of the window to ~ 60 Hz near the merger, in agreement with the prediction of eq. (28) that the dynamical part of $P^{(0)}$ oscillates at twice the orbital frequency. The amplitude grows monotonically as the binary inspirals, in qualitative agreement with the scaling implied by eq. (28). The vanishing of the time average of ΔE, evident from the figure, is the numerical counterpart of eq. (33).

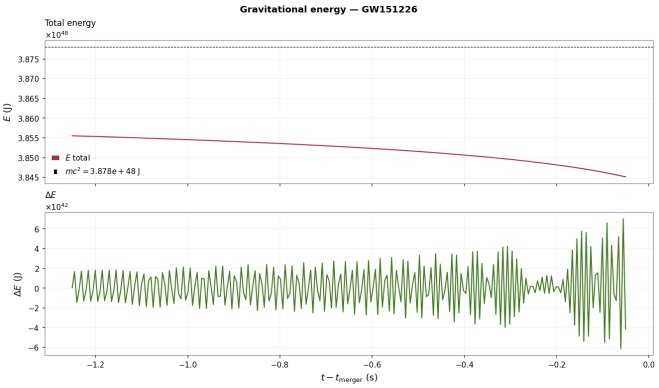

FIGURE 3: Top: total gravitational energy $E(t) = cP^{(0)}(t)$ of the GW151226 space-time, obtained from the numerical evaluation of eq. (27) on a cubic surface of side $L = 440$ Mpc. The dashed line indicates the ADM mass-energy $mc^2 = 3.878 \times 10^{48}$ J. The curve shows the sum $mc^2 + E_b(t)$ given by eq. (31); the radiation contribution $\Delta E(t)$ is invisible at this scale. Bottom: gravitational-radiation contribution $\Delta E(t) = E(t) - mc^2 - E_b(t)$, obtained from the terms of the integrand of eq. (27) that involve the temporal derivatives of the metric tensor, displayed at its own scale of 10^{42} J. The oscillation frequency tracks $f_{\rm gw}(t)$, the amplitude grows along the inspiral, and the time average vanishes, all in agreement with eqs. (28) and (33).

6. CONCLUDING REMARKS

In this paper we have analyzed the metric tensor for the nonspinning black hole binary in circular orbit in the xy plane, in the context of the teleparallel equivalent of general relativity. This metric tensor is an approximate solution of Einstein's equations in which the distance between the holes is constant in time.

The calculations of the total gravitational energy, and the corresponding flux for the nonspinning black hole binary in circular orbit, yield a quite interesting result. The total energy $P^{(0)}$ and the gravitational flux $\Phi^{(0)}$ are given by eqs. (28) and (32). The former yields the known result for the mean value of

the binding energy of the configuration, whereas the average value of $\Phi^{(0)}$ in time vanishes, $< \Phi^{(0)} >= 0$. It means that for two black holes in circular orbit, the average value of the gravitational radiation is zero. This result is consistent with the stationary character of the space-time. A nontrivial emission of gravitational radiation must necessarily be related to a loss of energy-momentum of the source (as the loss of mass described by Bondi's radiating metric [12]). If, however, the separation between the holes decreases in time, as in an actual evolution of the black hole binary, a nonvanishing (definite positive) flux of gravitational radiation is emitted. The vanishing time average obtained for the stationary circular configuration refers to the gravitational-field flux evaluated in the vacuum radiation zone of the approximate binary-black-hole spacetime considered here. It should not be identified directly with the source-based quadrupole energy-loss formula, whose observational success in binary-pulsar systems concerns the secular evolution of material compact binaries. A TEGR analysis of eccentric binaries with matter sources requires a different spacetime model and is beyond the scope of the present work.

In section 5 we have illustrated the analytical results of section 4 with a concrete application to the event GW151226. Using the strain data of the L1 detector, we have reconstructed the time-dependent orbital parameters $b(t)$, $\omega(t)$ and $\dot{b}(t)$ from the fitted inspiral waveform, and evaluated the gravitational energy $E(t) = cP^{(0)}(t)$ from eq. (27) by direct numerical integration over a cubic surface of side $L = 440\,\mathrm{Mpc}$. The total energy is dominated by the ADM mass-energy mc^2 and is reduced below this value by the binding energy of eq. (31), which tracks the shrinking orbital separation. The structure of the integrand of eq. (27) naturally splits the total energy into a slowly-varying piece dominated by the spatial derivatives of the metric, and a rapidly oscillating piece coming from the temporal derivatives, the latter constituting the genuine gravitational-radiation contribution to $P^{(0)}$. Both contributions are quantitatively consistent with eqs. (28), (31), (33) and (34), confirming that the formalism developed in sections 2 and 4 admits a meaningful numerical realisation on actual gravitational-wave data.

This result is conceptually different from the conclusion drawn from Eddington's spinning rod [29], which was reconsidered by other authors [30]. In the framework of pseudotensor definitions and of linearized general relativity, the quadrupole formula was obtained. The latter relates the energy loss of the system with the variation in time of the mass-quadrupole of the source. This formula was used by Eddington to deduce the energy flux generated by a rod that spins in the xy plane with angular frequency ω. Let I represent the moment of inertia of the rod, and G the gravitational constant. The total energy flux is given by [29, 13]

$$\frac{dE}{dt} = -\frac{32GI^2\omega^6}{5c^5}\,. \tag{37}$$

In contrast to the approach that allows one to deduce the formula above, we note that the procedure that led to eqs. (32) and (33) is based neither on pseudotensors nor on the linearized form of the theory. Equation (10) is a true tensorial quantity. It is valid for any coordinate system of the three-dimensional spacelike hypersurface, and for finite volume V and corresponding surface S. We believe that the reconsideration of Eddington's spinning rod in the present context would also lead to an equation of the type $< \Phi^{(0)} >= 0$, provided the angular frequency is constant.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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